A theoretical analysis of the cost structure of urban logistics

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Abstract. This paper analyses with an analytical microeconomic model how warehouse location depends on the other economic parameters of urban logistics, such as transport cost, freight demand, and rents. In the model, a warehouse located at the outskirts of a city dispatches trucks towards the city center in order to deliver and pickup commodities. A logistic cost function is calculated, consisting in the transport costs and the land use cost, which both depend on the distance of the warehouse from the city center. The model shows that among the possible causes of logistic sprawl, an increased demand in deliveries in city centers can also explain why warehouses tend to locate further and further from city centers. The model also shows that the cost function exhibits economies of scales. Potential implications in terms of public policy are discussed.

Keywords: “logistics sprawl”, “warehouse location”, “land use”, “microeconomics”

1 Introduction

Urban logistics is an essential function of cities: it makes commodities and services available to city inhabitants. It operates under strong constraints: consumers have high expectations in terms of level of service, whereas the impacts of urban logistics (congestion, emissions, etc.) should be kept to a minimum. As a consequence, urban logistics is often handled in an ambivalent, and not always strategically consistent, way by city officials.

One of the most important challenge in urban logistics is to deliver quickly many small shipments to many different customers, be they individuals, retail stores, or firms. Distribution centers, i.e. warehouses dedicated to transhipments, with little to no storage, are instrumental in achieving this performance. Shipments with spatially similar destinations are carried together in large vehicles outside cities, transshipped through distribution centers, and finally delivered by small vehicles during rounds to their receivers: distribution centers operate the junction between interurban logistics and urban logistics.

The location of a distribution center is a complex decision, influenced by many factors. Among other issues of public policy, the closer a distribution center is from a city center, the less heavy goods trafic; but the less land is available to other uses. One would expect that with the increased sensitivity of cities to heavy goods trafic, the development of e-commerce, the expected substantial increase of energy costs in the long term, and so on, warehouses would locate closer to city centers, or at least not too much farther. The contrary is observed instead: warehouses tend to locate further and further; furthermore, it seems that warehouses are moving faster than city size: their movement is not only accompanying urban growth.

This phenomenon, known as logistics sprawl, raises several questions: what are its causes? Is it due to the dynamics of land use prices alone or to the evolution of other parameters? What are the impacts? Is there cause for concern and for political action? These questions have already been the object of many contributions, which will be discussed in Section 2. The objective of this paper is to provide some additional insight by developing a relatively simple analytical model. The model is designed so as to represent warehouse location as a tradeoff between rental costs and transport costs (Section 3). A simple case is considered, where shipments are processed through a warehouse at the periphery of a given city and then transported towards the city center. The logistic cost function is written; the cost-minimising warehouse location is then derived. The influence of the model's parameters on its outputs is then analysed in Section 4, as well as the structure of the supply chain's costs. First implications in terms of public policy are discussed. By construction, the approach presented in this paper is necessarily oversimplifying; the resulting limitations are discussed together with the paper's conclusion in Section 5.

2 Literature review

Urban logistics is a complex system [16, 12]. Inside that system, warehouse location is a complex process [14, 13]. Warehouse location should be optimal for the performance of the supply chain or supply chains in which it takes place. Optimality does not mean the same thing in different supply chains, and therefore in terms of warehouse types, but in general transport costs will have a significant weight. For given, known transport flows with a given, unique unit transport cost, the position minimizing transport costs is the geometric median. However, at the scale of the expected lifetime of a given warehouse, transport flows are often unknown; as a consequence developers will prefer locations within important freight corridors, and also proximity to infrastructure nodes. Furthermore, other factors, such as the availability of adequate land parcels, rent costs, and the nature and easiness of the negotiations with the public authorities also play important roles. Finally, in the case of distribution centers operating the junction between interurban and urban logistics, inward and outward shipments are carried by vehicles of different types, therefore different operating costs; this also influences the optimal warehouse location.
Despite this complexity, a general movement of logistics sprawl has been observed over the past few decades, in many major metropolitan areas. [7] synthesizes the current academic evidence concerning this phenomenon. In [7], logistics sprawl is defined as "the spatial deconcentration of logistics facilities and distribution centres in metropolitan areas." Logistics sprawl has been studied by 20 case studies and observed in 17 of these case studies. For example, [1] examines the relocation of parcel transport terminals in Paris between 1974 and 2010, and concludes that the average distance of these terminals from the city center increased to 18.1 km from 6.4 km, i.e. faster than the spatial growth of the metropolitan area of Paris. [15] studies the case of Tokyo, and identifies a similar, although weaker, trend. [15] is interesting as it presents an econometric warehouse location model; it also shows that rent does not influence warehouse locations *per se*, those depend on the rent gradient instead. The model designed in this paper yields similar conclusions. The North-American case is examined in [8] and [6], who identifies a similar trend in Los Angeles and Atlanta, but not in Seattle. The specific evolution of Seattle could be due to a unusually constraining land use policy (compared to the other case studies), although other explanations cannot be ruled out.
This review would not be relevant without addressing the freight and logistics modeling literature. Freight and logistics modeling is a wide domain, with approaches which can differ a lot depending on the objectives, data and resources (see e.g. [17, 4]). For the sake of illustration, one can quote statistical models for demand estimation (such as [11] or [10]), operations research for optimisation (see e.g. [2]), traffic engineering for the analysis of the interaction between urban logistics and urban congestion [3], or analytical, economic analyses [9] as hints of the range and depth of the field. This paper is in the line of [9]: it develops an analytical model, based on necessarily simplified assumptions, but designed so as to make economic analyses possible.

3 Model description

The objective of this section is to present a simple model of optimal distribution center location in a context relevant to urban logistics. The problem and assumptions are presented, then the cost function is calculated.

3.1 Problem description

In a simplified manner, one can consider that for a given shipment to be delivered in a given city, freight transport can be organized in three ways:

- 'one-to-one transport' a vehicle carries the shipment alone, from the shipper to the customer and then come back;

- 'one-to-many transport' a vehicle carries many shipments, from the shipper to a set of customers in the city and then come back;

- 'many-to-many transport' a heavy goods vehicle moves all the shipments due a certain day or week to a distribution center located near the city, then load them in smaller vehicles which then deliver them to the customers.

The third option is particularly relevant for small flows of fast moving goods [5]. Its main advantage is that it allows the simultaneous use of small vehicles and large vehicles in the same supply chain. However, it requires a building and the associated land/rent and transshipment costs.
Let us now set the assumptions of the model. Consider a shipper sending commodities to a city of area A from a plant or a national warehouse located far from that city. The supply chain between that plant and the customers in the city consists of a sequence of successive stages: the commodities are transported from that plant to a distribution center located at distance l from the city center (a). The commodities are unloaded in the distribution center and stored, and sorted, and then loaded into smaller vehicles (b); each of those vehicles makes a round, delivering several customers in the process. The round consists of an approach movement and a return movement (c), and all the intermediate movements inside the city, between each delivery (d).

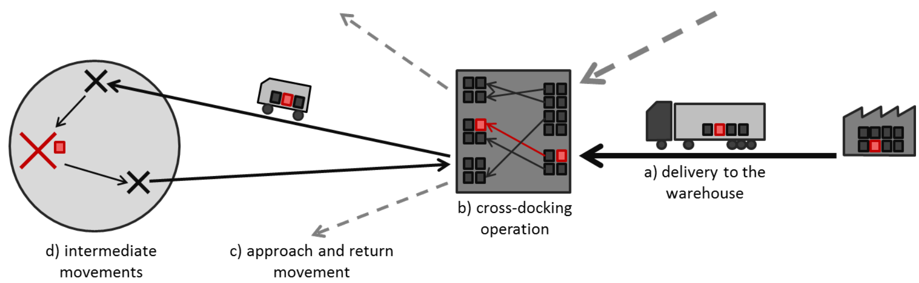

Figure 1: Description of the transport operation

The location of the distribution center in not neutral in terms of costs: the closer it is to the city center, the less transport costs on the urban side, but the higher rent costs. In order to identify the optimal distance to the city center, it is necessary to derive the shipper's logistic cost function.

3.2 Cost function

The total transport and transshipment costs consists of three components: the cost of carrying the commodities from the plant to the distribution center; the distribution center and transshipment costs; and the cost of carrying the commodities from the distribution center to customers. The first cost component is not critical in the context of this paper: the commodities are transported by heavy goods vehicles over typically long distances, and the corresponding cost does not depend much on the location of the distribution center or on what happens on the city side.
The second cost component consists of the fixed and variable cost of the distribution center C^w. Out of simplicity, let us assume that the distribution center's size is proportional to its throughput and that its rental and operating costs are both proportional to its size, and decreasing functions of the distance of the distribution center to the city center (on the basis of the radial symmetry assumption). Let $c_w(l)$ denote the distribution center and transshipment cost, in monetary unit per shipment.
The third cost component is the transport cost on the city side, or the last-kilometer transport cost, C^t. Each vehicle delivers multiple customers by making rounds. The more customers there is in a round, the less costly it is to deliver to each of them. However, round's lengths are constrained. In practice, several types of constraints, or regimes, can be distinguished. The following two are directly interesting in this research:

- Regime 1: the round's length is constrained by the duration of the driver's work day H. (H depends on whether deliveries are made in the morning and pickups in the afternoon or during the whole day).

- Regime 2: the round's length is constrained by the delivery lead time expected by the customers, and that delivery lead time is lower than the driver's work day duration. If customers expect to be delivered in less than H hours after ordering, the round's duration cannot[1] be larger than H.

These two regimes are not very different from the perspective of the supply side: in both cases, the round duration cannot exceed H. The main distinction will be on the demand side; under regime 2 round duration H is determined by the preferences of customers. Two other regimes could also be distinguished: one where the round's length is constrained by the capacity of the vehicle; the other where the round's length is constrained by the vehicle's autonomy.

For the time being, consider the number of shipments to be delivered during a period of duration H as given and equal to F. Then, the length of each round will depend on the length of the approach and return movements, approximately l, and also on the distance between two consecutive customers in the city center. Let δ denote this distance. It is very difficult to compute exactly, as finding the rounds that minimise the transport cost is a NP-hard problem. However, it can be approximated. Intuitively, if the area of the city is multiplied by 4 and the number of customers to deliver is unchanged, then δ is only multiplied by 2. In general, in a uniform area δ is equal to:

$$\delta = k\sqrt{\frac{A}{F}}, \tag{1}$$

with k a positive constant [9].

Now, denote by v_a (resp. v_z) the vehicle speed outside (resp. inside) the city center. Consider a vehicle delivering x customers. The round's duration is then:

$$d = 2\frac{l}{v_a} + x\left(h + \frac{\delta}{v_z}\right); \tag{2}$$

where h is the time a delivery actually takes, or the idle time for the vehicle at each customer's, and x is unknown. Given that $d = H$, $x = (H - 2l/v_a)/(h + \delta/v_z)$. This requires that $H > 2l/v_a$. Let $H_r = H - 2l/v_a$ denote the duration of the round excluding the duration of the approach and return movements; and let $h_o = h + \delta/v_z$ denote the duration of an operation including the travel time between two consecutive customers. Then: $x = H_r/h_o$. Denote by L the average length of a round: it is then $L = 2l + \delta x$, or $L = 2l + \delta H_r/h_o$. Denote by R the average number of rounds necessary to deliver the F customers. It is then $R = F/x$, or $R = Fh_o/H_r$.

Eventually, transport costs depend on the vehicle operating cost on one hand, and the workforce and vehicle capital cost on the other hand. Vehicle operating cost is assumed to be proportional to distance, up to a coefficient c_l. Workforce and vehicle capital costs are assumed to be proportional to total round duration, up to a coefficient c_h. The transport cost function is then $C^t = c_lRL + c_hHR$. Let $c_R = 2c_ll + c_hH$ denote the "fixed" part of the transport cost, i.e. the part which does not depend on δ. By making explicit all the terms of the transport cost function and adding the platform cost function, the total cost function becomes (see Appendix):

$$C = \left(c_w + c_R\frac{h_o}{H_r} + c_l\delta\right)F. \tag{3}$$

By replacing δ with Equation (1), Equation (3) becomes (see Appendix):

$$C = \left(c_w + \frac{c_R}{H_r}h\right)F + \left(\frac{1}{v_z}\frac{c_R}{H_r} + c_l\right)k\sqrt{AF}. \tag{4}$$

In the following, let C_y denote the partial differential of C with respect to variable y.

Equations (3) and (4) reveals the importance of density in the economics of urban logistics, with implications in terms of cost structure and regulation. Those are further analysed in Section 4.2. Now, observe that Equation 6 depends on the distance between the city center and the platform. The immediate question is whether there is a distance for which the total cost is minimal and what is its value.

[1]This is a simplifying assumption. It may be – theoretically – possible for the shipper to dispatch before customers actually order, thus anticipating the delivery process, given adequate forecasting technologies. However technically difficult this may seem, the paper will hopefully show the extremely strong financial appeal of such an organisation.

4 Theoretical analyses

In this section, the equation of the optimal distribution center location is first derived, then analysed. Then, the structure of the cost function is analysed, and some implications are discussed.

4.1 Optimal location of the distribution center for a given demand

The optimal distance is defined here as the distance which minimises the total cost for a fixed demand. Let us assume that c_w is a differentiable function of l. Then, if the optimal length is not a border solution, a necessary condition for l to be optimal is that $C_l = 0$ or, equivalently:

$$Fc'_w + 2Fc_l \frac{h_o}{H_r} + 2Fc_R \frac{h_o}{H_r^2 v_a} = 0. \tag{5}$$

At the optimal location, the additional cost of being one kilometer closer to the city center should be equal to the travel cost it saves. The travel cost savings come from the reduced length of the approach and return movements and from the fact that less rounds are necessary to deliver F customers when the distribution center is closer to the city center. Note that the optimal location appears to be independent on F; it doesn't: the density parameter δ, which depends on F, influences both h_o and H_R.
The distribution center's optimal location does not depend on the rental costs gradient, but on its *variation*. The willingness of the shipper to pay to reduce the distribution center's distance to the center increases with l: the impact of l on the number of rounds is stronger when l is large. If the variation of the rental cost is constant or decreasing with l (i.e. if the rental curve is decreasing and convex), then the optimal location is necessarily unique. If the rate is convex and decreasing fast enough, then the solution is unique; but in general many configurations are possible.
In order to analyse how the optimal distribution center location depends on the model parameters, the implicit equation theorem is applied. The calculations (see Appendix) allow to conclude that in general:

- the optimal distance decreases with c_l and c_h: when the kilometre cost or the hour cost of operating a vehicle increases, transport costs more; it is then profitable to be closer to the city center.

- the optimal distance decreases with h: the slower the delivery and pickup operations, the larger their share in the transport cost, the more profitable it is to bring the distribution center closer to the city center.

- the optimal distance increases with H: when the work's day duration increases, the relative weight of the approach and return movements decreases in the cost function. It is not necessary to pay as much to be close to the city center;

- the optimal distance increases with v_z and v_a: this effect is somewhat symmetric to that of h and H.

- the optimal distance decreases with δ, i.e. increases with F. When there are more delivery or pickup operations, they are denser in a given area. Each operation costs less; it is profitable to locate the distribution center further from the city center.

These results can be summarised as follows: if any of the cost parameter increases, the optimal distance decreases; if any of the speed parameter increases, the optimal distance increases; if the unit cost of a delivery operation increases, due to a longer delivery duration or to a lesser delivery location density, the optimal distance decreases; if the work day constraint H increases, the optimal location increases; if the average distance δ increases, the optimal location decreases. Symmetrically, when the optimal distance increases with the total demand. Indeed, the more customers for a given warehouse, the higher the density of delivery locations, hence the unit cost decreases and the benefit to be close to the city center is not as large. In other words, the mere increase in the number and frequency of deliveries in cities might have contributed directly to logistic sprawl.

4.2 Analysis of the cost structure of the supply chain, possible implications

The cost structure of the supply chain and the consequences in terms of market structure and distortion depend directly on the relationship between F and C. If δ were fixed then the cost function would be linear; there would be no returns to scale[2]; and thus no ground for regulation[3], and the marginal cost and the average cost would be equal. However, δ is not fixed, and the marginal cost of an additional customer in the time period is, based on Equation (4):

$$C_F = c_w + \frac{1}{2}c_l\delta + c_R\frac{h_o - \delta/2v_z}{H_r}. \tag{6}$$

It can be compared to the average cost which is:

$$C_F = c_w + c_R\frac{h_o}{H_r} + c_l\delta. \tag{7}$$

The marginal cost is lower than the average cost: the cost structure of the supply chain of this model exhibits increasing returns to scale. The difference between the marginal cost and the average cost is the marginal external cost of an additional customer.

$$\frac{C}{F} - C_F = -\frac{1}{2}c_l\delta - \frac{1}{2}c_R\frac{\delta/v_z}{H_r}. \tag{8}$$

This difference has two causes: first, an increase in density reduces the distance between two successive operations inside the city center, and thus the corresponding cost. Second, this also allows to deliver more locations during the same time. The negative marginal external cost is exactly symmetric to the positive marginal external cost of congestion in road transport. Not only does an additional driver on a given road suffers from the traffic jam; by contributing to the traffic jam it also inflicts on all the other drivers a marginally increased travel time. In the case of urban logistics it is the contrary: an additional customer will cost less than the average cost because it has good chances to be on the way between two existing customers.

From the perspective of market structure, whichever way the market works, equilibrium prices will always be larger than marginal costs, without regulation (i.e. subsidies). As a consequence, the demand will be lower than it would be at its optimal level[4]. This is a typical market failure, which calls for a correction. This correction can take many forms. The most obvious, but perhaps the least realistic one is that of a direct subsidy to urban logistics. Other indirect means can be considered, such as urban planning options. A possible urban planning option would be to assign land to logistic premises. This raises complex issues, such as how and where this should be done, and also whether and how the resulting land use prices should be controlled, to actually correct the imperfection. Those questions are still open.

5 Conclusion

This paper presents a simple analytical model of the cost structure of urban logistics. The paper's main result is the relationship between the land use market, the parameters of urban logistics (vehicle speed, workday duration, costs and the demand) and the location of a warehouse. The paper shows that an exogenous increase in the demand for pickup and delivery operations results in warehouses locating further from cities, as a result of transport operations weighing relatively less in their cost function.

In addition, the paper briefly sketches some possible implications in terms of market structure and equilibrium pricing. Even accounting for the limitations of the model, urban logistics is most probably characterised by economies of scales. This results in non trivial conclusions in terms of market efficiency, and also of the policy actions that could or should be taken to address potential market imperfections.

[2] to the extent where several strong assumptions hold, including the constant returns to scale of the distribution center costs.

[3] aside from the regulation of all the transport externalities: congestion, accidents, noise, pollution and GHG emissions, for which adequate instruments exist, and could theoretically be applied.

[4] this optimal level is the one that would be obtained by a 'benevolent planner' or, perhaps a little more realistically, if all inhabitants planned together the location of warehouses

At this stage, this research is very much a work in progress. Many directions shall be explored in the short term: the various regimes discussed in the introduction will be analysed; the implications in terms of market structure and potential ground for the implementation of policy measures shall also be closely examined. From a technical perspective, it is necessary to extend the model to an endogenous demand, where the amount of pickup and delivery operation derives from the price charged by shippers. This demand could be generic, or could derive from models of inventory theory. The second option, while it needs an empirical validation, is useful insofar as it explains why pickup and delivery frequency has value to the inhabitants of the city center.
This work will leave many open questions for the medium to long term. One particularly important question, both technically and empirically, is about how does the urban logistics market work. Markets with economies of scales are complicated to study; they are often characterised by monopolistic competition. In our context, a particular question is whether this competition is spatial or not. In other words, do various logistic service providers and carriers compete in the same areas in the city center or do they tend to locate in distinct territories? The answer depends most probably on the market segment, and data will be needed at some point to explore the question further. The second important question pertains to the externalities of urban freight transport: congestion, pollution, noise, and accidents. It should be studied together with the policy instruments already in place to address them. Precisely, the question is: to what extent do the conclusions obtained in this study still hold when these externalities are accounted for?

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Appendix

Derivation of Equation (3) The extended formula of the cost function is: $C = c_w F + (2c_l l + c_h H)\times F(h+\delta/v_z)/(H-2l/v_a) + c_l \delta F$. Replacing δ by Equation (1) gives: $C = c_w F + (2c_l l + c_h H)\times\ (hF + k\sqrt{AF}/v_z)/(H-2l/v_a) + c_l k\sqrt{AF}$. Equivalently: $C = (c_w + h(c_l l + c_h H)/(H-2l/v_a))F + ((2c_l l + c_h H)\times (v_z(H-2l/v_a)) + c_l)k\sqrt{AF}$.

Derivation of Equation (5) Let us differentiate the cost function C with respect to l. Remind that $dc_R = 2c_l dl$, and $dH_r = -(2/v_a)dl$. Then: $C_l dl = c'_w F dl + 2c_l F(h_o/H_r)dl + c_R h_o/(H_r^2 v_a)F dl$. If l is an optimal interior solution, then $dC = 0$.

Derivation of Equation (6) Let us differentiate the cost function C with respect to F. Remind that $dh_o = d\delta/v_z$ and that $d\delta = -(\delta/2F)dF$. Then $dC = c_w dF + c_R(dh_o)/(H_r)F + c_R(h_o)/(H_r)dF + c_l d\delta F + c_l \delta dF$. Or $dC = c_W dF - ((c_R)/(H_r v_z) + c_R)(F\delta)/(2F)dF + (c_R h_o/H_r + c_l\delta + c_w)dF$. Finally: $dC = c_w dF + (c_l\delta/2)dF + (c_R)/(H_r)(h_o - \delta/2v_z)dF$. Note that $h_o = h + \delta/v_z$ so that $h_o - \delta/2v_z = h + \delta/2v_z$.

Optimal location of the distribution center for a given demand: comparative statics The total differential of the cost function is $\sum_{y\in Y} C_{ly} dy$, where Y is the set of the parameters of C. Let us divide it by F and denote it by Γ. Then $\Gamma = c''_w(l)dl + (2h_o/H_r)dc_l + 2c_l dh_o/H_r - (2c_l h_o^2/H_r^2)dH_r + (2h_o/H_r^2 v_a)dc_R + (2c_R/H_r^2 v_a) - (4c_R h_o/H_r^3 v_a) - (2c_R h_o/H_r^2 v_a)dv_a$. Remind that the total differential of h_o is $dh_o = dh + 1/(v_z)d\delta - \delta/(v_z^2)dv_z$ and that the total differential of H_R is $dH_r = dH - (2/v_a)dl + (2l/v_a^2)dv_a$ and the total differential of c_R is $dc_R = 2l dc_l + 2c_l dl + H dc_h + c_h dH$.
It is now possible to analyse the behaviour of the optimal location of the warehouse. Analysing the second derivatives of C is analogous to analysing the first derivatives of Γ.

$$\Gamma_l = c''_w + \frac{8c_l h_o}{H_r^2 v_a} + \frac{2c_R h_o}{H_r^2 v_a}$$

If the rent is a convex function of the distance to the center, then c''_w is positive. Besides, in the vicinity of an equilibrium point, c''_w should be positive. As a consequence: $\Gamma_l > 0$. The other differentials are as follows:

$$\Gamma_{c_l} = \frac{2h_o}{H_r} + \frac{4lh_o}{H_r^2 v_a} > 0 \ \text{ and } \ \Gamma_h = \frac{2c_l}{H_r} + \frac{2c_r}{H_r^2 v_a} > 0 \ \text{ and } \ \Gamma_\delta = \frac{2c_l}{H_r v_z} + \frac{2c_R}{H_r^2 v_a v_z} > 0$$

$$\Gamma_{v_z} = -\frac{2c_l\delta}{H_r v_z^2} - \frac{2c_R\delta}{H_r^2 v_a v_z^2} < 0 \ \text{ and } \ \Gamma_H = -\frac{2c_l h_o}{H_r^2} + \frac{2c_h h_o}{H_r^2 v_a} - \frac{4c_R h_o}{H_r^3 v_a}.$$

Note that $(2c_h h_o/H_r^2 v_a) - (4c_R h_o/H_r^3 v_a) = (1/H_r^2 v_a)(2H_r c_r h_o - 4c_R h_o)$ and, given the fact that $c_R = 2c_l l + c_h H$ and $H_r = H_2 l/v_a$, that $2H_r c_r h_o - 4c_R h_o = 2H_r c_h h_o - 8c_l l h_o - 4c_h H h_o = 2(H_r - H)c_h h_o - 8c_l l h_o - 2c_h H h_o \leq 0$, so that $\Gamma_H < 0$. Finally:

$$\Gamma_{v_a} = -\frac{4c_l h_o l}{H_r^2 v_a^2} - \frac{8c_R h_o l}{H_r^3 v_a^3} - \frac{2c_R h_o}{H_r^2 v_a} < 0 \text{ and } \Gamma_{c_h} = \frac{2h_o H}{H_r^2 v_a} > 0.$$