A bilevel uncapacitated location/pricing problem with Hotelling access costs in one-dimensional space

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Abstract. We formulate a spatial pricing problem as bilevel non-capacitated location: a leader first decides which facilities to open and sets service prices taking competing offers into account; then, customers make individual decisions minimizing individual costs that include access charges in the spirit of Hotelling. Both leader and customers are assumed to be risk-neutral. For non-metric costs (i.e., when access costs do not satisfy the triangle inequality), the problem is NP-hard even if facilities can be opened at no fixed cost. We describe an algorithm for solving the Euclidean 1-dimensional case (i.e., with access cost defined by the Euclidean norm on a line) with fixed opening costs and a single competing facility.

Keywords: "facility location", "bilevel programming", "Stackelberg games" .

1 Introduction

Starting with the seminal work of [7], spatial competition models have been a fruitful setting for analyzing price competition in oligopoly situations. In the majority of these models, conditions for the existence of a price equilibrium between two companies competing for customers distributed on a line are sought. Several variants of these models have been analyzed; see e.g., [1, 4, 5, 8, 10, 12, 13, 14, 15] for earlier and more recent contributions, and the references therein. In this paper, we analyze a two-level location/pricing problem for deciding facility sites and service prices for a firm maximizing its utility and facing competition from a single competitor at the first stage. In the second stage, the customers make their decisions of choosing the facilities where they will be serviced. In the present paper, we are not concerned with potential reactions of the competitor. Hence, we shall not study equilibrium issues, rather we shall limit ourselves with the optimal facility-opening and pricing decisions of the leader company.
Let $F, |F| = m$, be a set of potential sites where a company already holds or can place a facility to sell a product or a service. Let also $C, |C| = n$, be a discrete set of customers potentially interested in being serviced from the facilities opened in F.
A *bilevel facility location problem* is a facility location problem where decisions are made according to a Stackelberg game with a leader (the company) and a set of followers (the customers). The leader makes its decision first — e.g., decides which facility to open and/or at which price to offer the service. Then the followers make their own decision, according to local convenience. Multi-echelon, multi-service, finite capacities can be envisaged as in classical location problems. The problem has potential applications in sectors where a company [1] is considering candidate locations for opening a new service point as well as a pricing policy for attracting clients from an existing competitor.
User preference is a quite general way of disciplining followers' behaviour, see [6]: each customer holds a list that ranks the sites of F from the most to the least preferred, and after the leader's decision prioritizes

[1] Most businesses such as fast food, spare parts, supermarket or dry cleaners chains, as well as pizzerias, diners, plumbers, hardware stores, gardeners, contractors, car washers can be considered as potential examples.

the facility with the highest rank among those opened. Preference can be of any type: if defined after a measurable parameter d_{ik} that depends on customer i (e.g., the cost of reaching the facility) then we speak of *measurable preference*. In this case, the list of customer i ranks the facilities from the smallest to the largest d_{ik} (with a specified tie-breaking rule in case of equally preferred facilities). A measurable preference is *metric* if d_{ik} fulfills the metric axioms:

- non-negativity: $d_{ik} \geq 0$ for all $i \in C$ and $k \in F$;
- identity: $d_{jj} = 0$ for all $j \in C \cup F$;
- symmetry: $d_{ik} = d_{ki}$ for all $i \in C$ and $k \in F$;
- subadditivity[2]: $d_{ik} \leq d_{ij} + d_{jk}$ for all $i, j, k \in C \cup F$.

Those axioms are often fulfilled by such measures as the reaching cost. More in general, the d_{ik} can be derived from any norm $\|.\|$ applied to a vector pair $\mathbf{r}_i, \mathbf{f}_i \in I\!R^p$: $\mathbf{r}_i$ describes user requirements, $\mathbf{f}_k$ service features, and $d_{ik} = \|\mathbf{r}_i - \mathbf{f}_k\|$ measures the distance between service offered and user expectations.
In the problems considered here, no preferred order among facilities is a-priori given, and we instead assume access costs as in Hotelling [7]. In other words, we suppose that the d_{ik} measure the cost customers must add to service price π_k in order to reach the facility. Thus, follower i accords its preference to the facility k that minimizes, among those opened by the leader, the sum $\pi_k + d_{ik}$. Of course, if the leader's decision includes the prices π_k at which service is offered in facilities $k \in F$, the follower's rank is not known in advance, but depends on the leader's choice. This observation marks a substantial difference with the user-preference location models proposed in the literature.
We note that, in a similar vein to our setting, Marcotte and co-authors [9, 11] have made contributions to the literature on bi-level optimization and Stackelberg games, where the leader is a firm and the clients are its followers. It is also possible that some of the clients are firms themselves.

2 Problem definition

Consider a two-level non-capacitated facility location problem where:

1. The leader decides (where to open facilities in F and) at which price the service can be offered to followers in each facility;
2. Each follower then decides whether to access or not service from a facility placed in k according to a measurable preference.

The leader's decision variables are

x_k: binary, set to 1 if site k is chosen for opening a facility, 0 otherwise;

π_k: real, equal to the price applied to each customer for accessing facility k.

Let c_k be the cost of opening a facility at $k \in F$. Also, for any follower $i \in C$ and site $k \in F$, let c_{ik} (let d_{ik}) denote a known third-party cost the leader (follower i) has to bear if i is serviced from the facility opened in k. It is assumed that the leader will not open a facility at exactly the site of the competitor. This assumption is in accordance with the results of Fischer [5] where it was shown that players are not to share facility sites in a duopolistic competitive location/pricing game in order to maximize profits.
Once the leader has made a choice, followers behave according to a greedy criterion, that is

Assumption 1 *Customer i chooses the facility k with the least total cost $\pi_k + d_{ik}$.*

The leader's problem is to open facilities and decide prices so as to maximize total utility. The trivial case of unbounded utility is ruled off by adopting the following

Assumption 2 (No-monopoly) *Any customer $i \in C$ can get service from a competitor at total cost $d_i = \pi_0 + d_{i0}$.*

[2]In the computer science and operations research community, subadditivity is usually referred to as the "triangle inequality".

Competitors can be located in different sites and offer services at different prices. However, in the general non-metric case we lose no generality by representing the competitor as a single dummy facility 0. In this case we let $F_0 = F \cup \{0\}$.
Unlike typical spatial competition models [3], the leader is assumed to know in advance whether a price π_k is or not convenient for customer i: that is,

Assumption 3 *The leader's knowledge of customer distances and competitors' prices is complete.*

The following assumptions regulate tie-breaking:

Assumption 4 *If $d_{ij} < d_{ik}$ and facilities at $j,k \in F$ offer service to i at the same total cost $\pi_j + d_{ij} = \pi_k + d_{ik}$, then customer i prioritizes j.*

Assumption 5 *If facility 0 and facility $k \in F$ offer service to i at the same total cost $\pi_0 + d_{i0} = \pi_k + d_{ik}$, then customer i prioritizes k.*

The rationale for Assumptions 4, 5 is that, in the cases foreseen, the leader can capture the clients assigned by a solution by discounting the price of an arbitrarily small amount.
Finally, we have the following assumption about the leader's and clients' utility functions

Assumption 6 *The leader and all clients are risk-neutral.*

Defining followers' (binary) variables y_{ik} for any customer i and any site $k \in F_0 = F \cup \{0\}$, we can state our problem as follows:

Problem 1 *Find values for $x_k \in \{0,1\}$ and $\pi_k \in \Pi_k \subseteq I\!R, k \in F$, that maximize the total utility*

$$U(\mathbf{x},\pi) = \sum_{k\in F} [\sum_{i\in C} (\pi_k - c_{ik})y_{ik} - c_k]x_k$$

of the leader, subject to

$$\min_{\mathbf{y}_i} \{d_i y_{i0} + \sum_{k\in F} (\pi_k + d_{ik})y_{ik} : \sum_{k\in F_0} y_{ik} = 1, y_{ik} \leq x_k\} \textit{ for all } k \in F_0\}$$

for all $i \in C$.

In the particular case of $c_k = 0\ \forall k$, Problem 1 is a mere price-setting problem. This assumption, however, does not simplify its solution:

Theorem 1 *Problem 1 is NP-hard even with $c_k = c_{ik} = 0$ for all $i \in C, k \in F$.*

Proof. By reduction from SET COVERING, see [2].

3 One-dimensional metric case with single competitor

This section treats the case where facilities and customers are located in a *metric space* and d_{ik} is the *distance* between customer i and facility k. The space is 1-*dimensional*, thus representable as a straight line r on which a reference (origin) is defined: call p_k the position (abscissa) of facility k, and q_i that of customer i. Therefore, the Euclidean distance between customer i and facility k is

$$d_{ik} = |q_i - p_k|\,.$$

Facility sites are assumed to be distinct, i.e., $p_j \neq p_k$ for $j \neq k$. Here, we limit our attention to a duopoly, that is, to problems with *just one competing facility* (note that metric properties are generally not retained when ascribing the facilities of competitors to a single dummy facility 0).
Without loss of generality, our notation will be such that:

- The competing facility is placed on the origin of the line, that is, $p_0 = 0$.

- The leader's facilities to the right (to the left) of 0 are numbered with positive (negative) integers, starting from 1 and increasing from left to right (from -1 and decreasing from right to left).

- Similarly, customers to the right (to the left) of 0 are numbered with positive (negative) integers, starting from 1 and increasing from left to right (from -1 and decreasing from right to left).

Thus, $j < k$ implies $p_j < p_k$ and $q_j \leq q_k$. Recalling (Assumption 1) that client i uses the facility that minimizes $d_{ik} + \pi_k$ in F_0, one can prove the following for any vector π of prices adopted by the facility:

Proposition 2 *Let $k \in F_0$, and $i < j$ be two customers that use facility k. Then, any customer h for which $i < h < j$ also uses facility k.*

Table 1: Sample problem with $|C| = 10$ and $|F_0| = 5$.

Facilities			**Customers**	
Index k	Position p_k	Price π_k	Index i	Position q_i
-2	-10	12	-5	-11
-1	-4	11	-4	-8
0	0	8	-3	-5
1	5	13	-2	-2
2	11	10	-1	-1
			1	2
			2	4
			3	6
			4	7
			5	8

Proposition 2 implies that, for any price vector, each facility will serve exactly one integer interval $[i, j]$ of customers on the line. This property is illustrated by the example of Table 1: facility -2 serves interval $[-5, -4]$; facility -1, interval $[-3, -3]$; facility 0 (i.e. the competitor), interval $[-2, 1]$. The remaining facilities 1 and 2 serve intervals $[2, 3]$ and $[4, 5]$, respectively. From now on, for a given price vector π, we will indicate as $[\lambda_k, \rho_k]$ (λ = left, ρ = right) the range served by facility k. The number of customers in the range $[\lambda_k, \rho_k]$ is given by $n_k = \rho_k - \lambda_k + 1$ for $\rho_k \lambda_k > 0$ and $n_k = \rho_k - \lambda_k$ otherwise.
As in the general case, the problem is to choose the facilities to open and the prices at which service is offered in each of them, so as to maximize the leader's total profit. To solve this problem, we use a result that expresses a property of *isolation at optimality* of a facility from the competitor:

Lemma 3 (Isolation) *Let facility* -1 *(respectively, facility* 1*) denote the first facility opened by the leader on the left (on the right) of the competing facility* 0*, and let $\pi \in I\!R^m$ be a vector of prices. If π is optimal, then any customer j with $q_j \geq p_1$ or $q_j \leq p_{-1}$ will be served by a facility owned by the leader.*

Proof (Sketch). Suppose indirectly that π is optimal and that a customer j located at $q_j \geq p_1$ is served by the competing facility 0. But (Assumptions 1 and 5) j served by 0 means $\pi_1 + d_{j1} > \pi_0 + d_{j0}$. Since $q_j \geq p_1$, the distance between j and facility 0 is $d_{j0} = p_1 + d_{j1}$, hence the cost inequality is rewritten

$$\pi_1 > \pi_0 + p_1. \tag{1}$$

Inequality (1) implies that, according to the optimal price vector π, facility 1 serves no clients. But reducing π_1 to $\pi_1' = p_1$, customer j will instead use facility 1, so increasing the leader's profit by at least p_1: this contradicts the optimality of π. Symmetrically, the reasoning is repeated for customers j for which $q_j \leq p_{-1}$.
In the following we will give an efficient method to solve the one-competitor one-dimensional problem with fixed costs of opening. To simplify exposition, we first consider the simple case in which fixed costs are identically null, then use the results obtained to tackle non-negative fixed costs.

4 The case $c_k = 0$

In this section, we describe an efficient algorithm for the case in which the cost c_k of opening facility k is 0 for any $k \in F$. We begin with two ground cases, where facility 0 is respectively placed to the *left* or to the *right* of all the sites in F.

Let us first look at the *left ground case*. Denote as r_k the index of the first customer to the right of $k \in F$, and consider the leftmost facility in F, that is 1. If the leader decides to set $\pi_1 = p_1$, facility 1 will serve every customer to its right at the same cost as 0, so the entire range $[r_1, n]$ will be served by 1 at price p_1. However, by reducing π_1 by $\Delta > 0$, one can enable facility 1 to serve customers to its left, so possibly increasing leader's profit. This decision not only involves the customers to the left of 1, but also all the prices in F, and a compromise is to be sought. For instance, with facility 2 coming into play, the leader can increase the gain by letting 2 cover the interval $[r_2, n]$ at a price higher than that fixed by facility 1: the corresponding gain is $\Delta U = (\pi_2 - \pi_1)(n + 1 - r_2)$.

In general, it is convenient to express the profits of the leader in terms of price differences between consecutive facilities. To this aim, define

$$\Delta\pi_k = \pi_k - \pi_{k-1}.$$

By the Isolation Lemma 3, the number of customers that use facility k is:

$$n_k = \lambda_{k+1} - \lambda_k,$$

where we write for uniformity $\lambda_{m+1} := n + 1$ and, if facility k serves no customer, $\lambda_{k+1} = \lambda_k$. With this notation, we can rewrite the objective function:

$$\sum_{k \in F} n_k \pi_k = \sum_{k \in F} (n + 1 - \lambda_k) \Delta\pi_k. \tag{2}$$

The generic term of the right-hand summation (2) only depends on π_k and on the index λ_k of the leftmost customer served by k. One can prove the following:

Theorem 4 *If π is a vector of optimal prices then $r_k \leq \lambda_k \leq r_{k-1}$.*

Thus we have a lower and an upper bound to the value that λ_k can take in an optimal solution. Now, for the largest possible value of $\Delta\pi_k$, two cases can occur depending on λ_k:

- If $\lambda_k = r_k$, price π_k must verify the restriction:

  $$\pi_k + q_{\lambda_k} - p_{k-1} \leq \pi_{k-1} + q_{\lambda_k} - p_k.$$

  Thus:

  $$\Delta\pi_k \leq p_k - p_{k-1}. \tag{3}$$

- If $r_{k-1} \leq \lambda_k < r_k$, the restriction on π_k to guarantee that k serves customer λ_k is:

  $$\pi_k + p_k - q_{\lambda_k} \leq \pi_{k-1} + q_{\lambda_k} - p_{k-1}.$$

  Thus:

  $$\Delta\pi_k \leq 2q_{\lambda_k} - p_k - p_{k-1}. \tag{4}$$

From Theorem 4 we also know that, in an optimal solution, the sets of values taken by λ_k, λ_j are mutually disjoint for $j \neq k$. Hence

Proposition 5 *The optimal values of $\Delta\pi_k$ do not bind one another, and can independently be determined from the positions of facilities and customers.*

To find an optimum price vector π^* it is then sufficient to calculate, independently for each facility $k \in F$, the index λ_k^* that maximizes the k-th term of the sum (2) where, according to (3), (4) :

$$\Delta\pi_k^* = \begin{cases} p_k - p_{k-1} & \text{if } \lambda_k^* = r_k \\ 2q_{\lambda_k} - p_k - p_{k-1} & \text{if } r_{k-1} \leq \lambda_k^* < r_k. \end{cases}$$

Optimal prices are then immediately derived as $\pi_k^* = \pi_{k-1}^* + \Delta\pi_k^*$ with initial value π_0^*, and the whole computation requires $O(m+n)$ steps. Prices are symmetrically computed in the *right ground case*.
In the more general case in which F contains both facilities on the left and on the right of 0, optimal prices can be separately computed for the left and right ground cases only if that $\pi_0 = 0$. To tackle $\pi_0 > 0$, it is useful to observe

Theorem 6 (Barrier) *Let* π^* *be an optimal price vector. Then no facility* $k > 1$ *(*$k < -1$*) will serve clients to the left (to the right) of* 0.

Using Theorem 6 we can easily find the optimum set of customers that lie in $[p_{-1}, p_1]$ and are captured by $-1, 1$. Then, combining the formulas for the left and right ground cases, we can again find π^* in $O(m+n)$ steps.

5 Extension to $c_k \geq 0$

The left (as well as the right) ground case with non-zero fixed costs can be solved by dynamic programming or, equivalently, as a longest (s,m)-path on an acyclic graph $G = (F \cup \{s\}, E)$, where F contains the candidate sites for the leader's facilities, and $jk \in E$ if $j = s$ and $k \in F$, or if $j,k \in F$ and $p_j < p_k$. Arcs are weighted by $w : E \to I\!R$ as follows:

$$w_{jk} = \begin{cases} \max\{p_k(n+1-r_k), f(k)\} - c_k & \text{if } j = s \\ \max\{(p_k - p_j)(n+1-r_k), g(j,k)\} - c_k & \text{if } k \in F \end{cases}$$

where

$$f(k) = \max_{i<r_k}\{(2q_i - p_k)(n+1-i)\} \qquad g(j,k) = \max_{r_j \leq i < r_k}\{(2q_i - p_j - p_k)(n+1-i)\}.$$

With an approach similar to Section 4, the two ground cases can be composed in order to find an optimal solution for the general case in which F has both elements on the left and on the right of 0. As a result, we can compute an optimal solution in $O(m^4 n)$ elementary steps.

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